Rankin-Cohen brackets for orthogonal Lie algebras and bilinear conformally invariant differential operators

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Abstract

Based on the Lie theoretical methods of algebraic Fourier transformation, we classify singular vectors of diagonally embedded generalized Verma modules for orthogonal Lie algebra and its conformal parabolic subalgebra with commutative nilradical, thereby realizing the diagonal branching rules. In particular, we construct the singular vectors for all one dimensional inducing representations, exploiting recursive structure for the generalized hypergeometric function ${}_3F_2$. As a geometrical consequence, we classify bilinear conformally invariant differential operators acting on homogeneous line bundles and realize their dependence on inducing representation theoretical parameters in an explicit way.

Key words: Generalized Verma modules, Diagonal branching rules, Rankin-Cohen brackets, Bilinear conformally invariant differential operators.

MSC classification: 22E47, 17B10, 13C10.

1 Introduction and Motivation

The subject of our article has its motivation in the Lie theory for finite dimensional simple Lie algebras applied to the problem of branching rules and composition structure of generalized Verma modules, and dually in the geometrical problem related to the construction of invariant bilinear differential operators or Rankin-Cohen-like brackets for automorphic forms associated to orthogonal groups.

The bilinear invariant differential operators, as the simplest representatives of multilinear invariant differential operators organized in an A_∞-homotopical structure, appear in a wide range of Lie theoretic applications. For example, the classical Rankin-Cohen brackets realized by holomorphic $SL(2,\mathbb{R})$-invariant bilinear differential operators on the upper half plane $\mathbb{H}$ are devised, originally in a number theoretic context, to produce from a given pair of modular forms another modular form. They turn out to be intertwining operators producing ring structure on $SL(2,\mathbb{R})$ holomorphic discrete series representations, and can be analytically continued to the full range of inducing characters. Consequently,

such operators were constructed in several specific situations of interest related to Jacobi forms, Siegel modular forms, holomorphic discrete series of causal symmetric spaces of Cayley type, etc., [5], [8], [16].

The main reason behind the underlying classification scheme for such class of operators is inspired by geometrical analysis on manifolds with, e.g., the conformal structure, and related PDE problems of geometrical origin. For $\tilde{M}$ a smooth (or, complex) manifold equipped with a filtration of its tangent bundle $T\tilde{M}$, $\mathcal{V}$ a smooth (or holomorphic) vector bundle on $\tilde{M}$ and $J^k\mathcal{V}$ the weighted jet bundle, a bilinear differential pairing between sections of the bundle $\mathcal{V}$ and sections of the bundle $\mathcal{W}$ to sections of a bundle $\mathcal{Y}$ is a sheaf homomorphism

$$B : J^k\mathcal{V} \otimes J^l\mathcal{W} \to \mathcal{Y}.$$

In the case $\tilde{M} = \tilde{G}/\tilde{P}$ is a generalized flag manifold, a pairing is called invariant if it commutes with the action of $\tilde{G}$ on sections of the homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$. Denoting $\mathbb{V}$, $\mathbb{W}$, $\mathbb{Y}$ the inducing $\tilde{P}$-representations of homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$, $\tilde{G}$-invariant differential pairings can be algebraically characterized as the space

$$Hom_{\mathcal{U}(\tilde{\mathfrak{g}})}(\mathcal{M}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}, \mathbb{Y}), (\mathcal{U}(\tilde{\mathfrak{g}}) \otimes \mathcal{U}(\tilde{\mathfrak{g}})) \otimes_{\mathcal{U}(\tilde{\mathfrak{p}}) \otimes \mathcal{U}(\tilde{\mathfrak{p}})} (\mathbb{V}^{\vee} \otimes \mathbb{W}^{\vee})). \qquad (1)$$

In other words, the former geometrical problem for finding bilinear invariant differential operators on $\tilde{G}/\tilde{P}$ acting on induced representations turns into a Lie algebraic problem of the characterization of homomorphisms of generalized Verma modules. In the geometrical context of flag manifolds and general curved manifolds with parabolic structure, a classification of first order bilinear differential operators for parabolic subalgebras with commutative nilradicals (so called AHS structures) was completed in [15]. One of the main applications of bilinear differential operators is that they act via invariant cup product as symmetries of invariant differential operators, see e.g. [4] for the case of conformally invariant Laplace operator.

Yet another approach to these questions is purely analytical and consists of meromorphic continuation of invariant distributions given by a multilinear form on the principal series representations. For example, a class of $G = SO_0(n+1, 1, \mathbb{R})$ (i.e., conformally)-invariant linear and bilinear differential operators was constructed as residues of meromorphically continued invariant trilinear form on principal series representations induced from characters, see [2].

To summarize, our article contains a general Lie theoretic classification of Rankin-Cohen-like brackets for the couple of real orthogonal Lie algebra $so(n+1, 1, \mathbb{R})$ and its conformal parabolic Lie subalgebra, and their explicit - in the sense of dependence on representation theoretical parameters - construction for characters as inducing representations.

The structure of our article goes as follows. As already mentioned, we first reformulate the existence of invariant bilinear differential operators (or equivalently, Rankin-Cohen-like brackets) in terms of purely abstract Lie theoretic classification scheme for diagonal branching rules of generalized Verma modules, associated to the real orthogonal Lie algebra $so(n+1, 1, \mathbb{R})$ and its conformal parabolic Lie subalgebra $\mathfrak{p}$. The reason behind the choice for this parabolic subalgebra is its fundamental property of having the commutative nilradical. Consequently, the branching

problem takes value in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}})$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$. Here the main device are character formulas and their reduction in the branching problems, see Section 2. The quantitative part of the problem consists of the construction of singular vectors, and is the content of Section 3. It is based on the procedure of rewriting the representation theoretical action in the Fourier dual picture, where the positive nilradical of $\mathfrak{p}$ is acting on symmetric algebra of the (commutative) opposite nilradical. This action produces the four term functional equation for singular vectors, and its solution is technically the most difficult part with both analytic and combinatorial aspects arising from generalized hypergeometric equation. The last Section 4 is a direct consequence of Sections $2, 3$ and determines the explicit formulas for bilinear conformally invariant differential operators representing these singular vectors.

Throughout the article we denote by $\mathbb{N}$ the set of natural numbers including zero.

2 Branching problem and algebraic Fourier transformation (F-method)

In the present section we briefly review the tool developed in [10], [11], which allows to explicitly realize the branching problem for the semisimple symmetric pair $(\mathfrak{g}\oplus\mathfrak{g}, diag(\mathfrak{g}))$ and the generalized Verma $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})$-modules induced from characters of the parabolic subalgebra $\mathfrak{p}\oplus\mathfrak{p} \subset (\mathfrak{g}\oplus\mathfrak{g})$. Throughout the article we use the notation $diag$ for a diagonally embedded subalgebra, e.g., $diag : \mathfrak{g} \hookrightarrow (\mathfrak{g}\oplus\mathfrak{g})$.

This approach is termed F-method and is based on the analytical tool of algebraic Fourier transformation on the commutative nilradical $\mathfrak{n}$ of $\mathfrak{p}$. It allows to find singular vectors in generalized Verma modules, exploiting algebraic Fourier transform and classical invariant theory. Its main advantage is the conversion of combinatorially complicated problem in the universal enveloping algebra of a Lie algebra into the system of partial or ordinary special differential equations acting on a polynomial ring.

Let $\tilde{G}$ be a connected real reductive Lie group with the Lie algebra $\tilde{\mathfrak{g}}$, $\tilde{P} \subset \tilde{G}$ a parabolic subgroup and $\tilde{\mathfrak{p}}$ its Lie algebra, $\tilde{\mathfrak{p}} = \tilde{\mathfrak{l}} \oplus \tilde{\mathfrak{n}}$ the Levi decomposition of $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{n}}_{-}$ its opposite nilradical, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_{-} \oplus \tilde{\mathfrak{p}}$. The corresponding Lie groups are denoted $\tilde{N}_{-}, \tilde{L}, \tilde{N}$. Denoting the fibration $p : \tilde{G} \to \tilde{G}/\tilde{P}$, $\tilde{M} := p(\tilde{N}_{-} \cdot \tilde{P})$ is the big Schubert cell of $\tilde{G}/\tilde{P}$ and the exponential map

$$\tilde{\mathfrak{n}}_{-} \to \tilde{M}, \quad X \mapsto \exp(X) \cdot o \in \tilde{G}/\tilde{P},\, o := e \cdot \tilde{P} \in \tilde{G}/\tilde{P},\, e \in \tilde{G}.$$

gives the canonical identification of the vector space $\mathfrak{n}_{-}$ with $\tilde{M}$.

Given a complex finite dimensional $\tilde{P}$-module $\mathbb{V}$, we consider the induced representation $\tilde{\pi}$ of $\tilde{G}$ on the space $Ind_{\tilde{P}}^{\tilde{G}}(\mathbb{V})$ of smooth sections of the homogeneous vector bundle $\tilde{G} \times_{\tilde{P}} \mathbb{V} \to \tilde{G}/\tilde{P}$,

$$Ind_{\tilde{P}}^{\tilde{G}}(\mathbb{V}) = C^{\infty}(\tilde{G}, \mathbb{V})^{\tilde{P}} := \{f \in C^{\infty}(\tilde{G}, \mathbb{V}) | f(g \cdot p) = p^{-1} \cdot f(g),\, g \in \tilde{G}, p \in \tilde{P}\}.$$

Let $\mathcal{U}(\tilde{\mathfrak{g}}_{\mathbb{C}})$ denote the universal enveloping algebra of the complexified Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Let $\mathbb{V}^\vee$ be the dual (contragredient) representation to $\mathbb{V}$. The generalized Verma module $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ is defined by

$$\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee) := \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\tilde{\mathfrak{p}})} \mathbb{V}^\vee,$$

and there is a $(\tilde{\mathfrak{g}}, \tilde{P})$-invariant natural pairing between $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ and $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$, described as follows. Let $\mathcal{D}'(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$ be the space of all distributions on $\tilde{G}/\tilde{P}$ with values in $\mathbb{V}^\vee$. The evaluation defines a canonical equivariant pairing between $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ and $\mathcal{D}'(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$, and this restricts to the pairing

$$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V}) \times \mathcal{D}'_{[o]}(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee \to \mathbb{C}, \tag{2}$$

where $\mathcal{D}'(\tilde{G}/\tilde{P})_{[o]} \otimes \mathbb{V}^\vee$ denotes the space of distributions supported at the base point $o \in \tilde{G}/\tilde{P}$. As shown in [3], the space $\mathcal{D}'_{[o]}(\tilde{G}/\tilde{P}) \otimes \mathbb{V}^\vee$ can be identified, as an $\mathcal{U}(\tilde{\mathfrak{g}})$-module, with the generalized Verma module $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$.

Moreover, the space of $\tilde{G}$-equivariant differential operators from $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ to $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V}')$ is isomorphic to the space of $(\tilde{\mathfrak{g}}, \tilde{P})$-homomorphisms between $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}'^\vee)$ and $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ for two inducing representations $\mathbb{V}$ and $\mathbb{V}'$ of $\tilde{P}$. The homomorphisms of generalized Verma modules are determined by their singular vectors, and the F-method translates searching for singular vectors to the study of distributions on $\tilde{G}/\tilde{P}$ supported at the origin, and consequently to the problem of solution space for the system of partial differential equations acting on polynomials $\mathrm{Pol}(\tilde{\mathfrak{n}})$ on $\tilde{\mathfrak{n}}$.

The representation $\tilde{\pi}$ of $\tilde{G}$ on $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ has the infinitesimal representation $d\tilde{\pi}$ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$, in the non-compact picture acting on functions on the big Schubert cell $\tilde{\mathfrak{n}}_- \simeq \tilde{M} \subset \tilde{G}/\tilde{P}$ with values in $\mathbb{V}$. The latter representation space can be identified via exponential map with $C^\infty(\tilde{\mathfrak{n}}_-, \mathbb{V})$. The action $d\tilde{\pi}(Z)$ of elements $Z \in \tilde{\mathfrak{n}}$ on $C^\infty(\tilde{\mathfrak{n}}_-, \mathbb{V})$ is realized by vector fields on $\tilde{\mathfrak{n}}_-$ with coefficients in $\mathrm{Pol}(\tilde{\mathfrak{n}}_-) \otimes \mathrm{End}\,\mathbb{V}$, see [13].

By the Poincaré–Birkhoff–Witt theorem, the generalized Verma module $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ is isomorphic as an $\tilde{\mathfrak{l}}$-module to the algebraic Weyl algebra of $\tilde{N}_-$-invariant of holomorphic differential operators on $\tilde{\mathfrak{n}}_-$, $\mathcal{U}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee \simeq \mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$. In the special case when $\tilde{\mathfrak{n}}_-$ is commutative, $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-)$ is the space of holomorphic differential operators on $\tilde{\mathfrak{n}}_-$ with constant coefficients. Moreover, the operators $d\tilde{\pi}^\vee(X)$, $X \in \tilde{\mathfrak{g}}$, are realized as differential operators on $\tilde{\mathfrak{n}}_-$ with coefficients in $\mathrm{End}(\mathbb{V}^\vee)$. The application of Fourier transform on $\tilde{\mathfrak{n}}_-$ gives the identification of the generalized Verma module $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$ with the space $\mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee$, and the action $d\tilde{\pi}^\vee$ of $\tilde{\mathfrak{g}}$ on $\mathrm{Diff}_{\tilde{N}_-}(\tilde{\mathfrak{n}}_-) \otimes \mathbb{V}^\vee$ translates to the action $(d\tilde{\pi}^\vee)^F$ of $\tilde{\mathfrak{g}}$ on $\mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee$ and is realized again by differential operators with values in $\mathrm{End}(\mathbb{V}^\vee)$. The explicit form of $(d\tilde{\pi}^\vee)^F(X)$ is easy to compute by Fourier transform from the explicit form of $d\tilde{\pi}^\vee$.

A generalization of the previous framework is based on $\tilde{P} \subset \tilde{G}$ introduced above, and yet another couple of Lie groups $\tilde{P}' \subset \tilde{G}'$ compatible with $(\tilde{G}, \tilde{P})$ in the sense that $\tilde{G}' \subset \tilde{G}$ is a reductive subgroup of $\tilde{G}$ and $\tilde{P}' = \tilde{P} \cap \tilde{G}'$ is a parabolic subgroup of $\tilde{G}'$. The Lie algebras of $\tilde{G}', \tilde{P}'$

are denoted by $\tilde{\mathfrak{g}}', \tilde{\mathfrak{p}}'$. In this case, $\tilde{\mathfrak{n}}' := \tilde{\mathfrak{n}} \cap \tilde{\mathfrak{g}}'$ is the nilradical of $\tilde{\mathfrak{p}}'$, and $\tilde{L}' = \tilde{L} \cap \tilde{G}'$ is the Levi subgroup of $\tilde{P}'$. We are interested in the branching problem for generalized Verma modules $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$ under the restriction from $\tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}'$.

Definition 2.1 *Let $\mathbb{V}$ be an irreducible $\tilde{P}$-module. Let us define the $\tilde{L}'$-module*

$$\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'} := \{v \in \mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee) |\, d\pi^\vee(Z)v = 0 \textit{ for all } Z \in \tilde{\mathfrak{n}}'\}. \tag{3}$$

Note that for $\tilde{G} \neq \tilde{G}'$, the set $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\mathfrak{n}'}$ is infinite dimensional completely reducible $\tilde{\mathfrak{l}}'$-module, while for $\tilde{G} = \tilde{G}'$ the set $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ is finite dimensional. An irreducible $\tilde{L}'$-submodule $\mathbb{W}^\vee$ of $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ gives an injective $\mathcal{U}(\tilde{\mathfrak{g}}')$-homomorphism from $\mathcal{M}^{\tilde{\mathfrak{g}}'}_{\tilde{\mathfrak{p}}'}(\mathbb{W}^\vee)$ to $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$. Dually, we get an equivariant differential operator acting from $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ to $\mathrm{Ind}^{\tilde{G}'}_{\tilde{P}'}(\mathbb{W})$.

Using the F-method, the space of $\tilde{L}'$-singular vectors $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ is realized in the ring of polynomials on $\tilde{\mathfrak{n}}$ valued in $\mathbb{V}^\vee$ and equipped with the action of $(d\tilde{\pi}^\vee)^F$.

Definition 2.2 *We define*

$$Sol(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}', \mathbb{V}^\vee) := \{f \in \mathrm{Pol}(\tilde{\mathfrak{n}}) \otimes \mathbb{V}^\vee |\, (d\tilde{\pi}^\vee)^F(Z)f = 0 \textit{ for all } Z \in \tilde{\mathfrak{n}}'\}. \tag{4}$$

Then the inverse Fourier transform gives an $\tilde{L}'$-isomorphism

$$Sol(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'; \mathbb{V}^\vee) \xrightarrow{\sim} \mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}. \tag{5}$$

An explicit form of the action $(d\tilde{\pi}^\vee)^F(Z)$ leads to a system of differential equation for elements in Sol. The transition from $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)^{\tilde{\mathfrak{n}}'}$ to Sol allows to transform the algebraic problem of computation of singular vectors in generalized Verma modules into analytic problem of solving the system of partial differential equations.

In the dual language of differential operators acting on principal series representation, the set of $\tilde{G}'$-intertwining differential operators from $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}(\mathbb{V})$ to $\mathrm{Ind}^{\tilde{G}'}_{\tilde{P}'}(\mathbb{V}')$ is in bijective correspondence with the space of all $(\tilde{\mathfrak{g}}', \tilde{P}')$-homomorphisms from $\mathcal{M}^{\tilde{\mathfrak{g}}'}_{\tilde{\mathfrak{p}}'}(\mathbb{V}'^\vee)$ to $\mathcal{M}^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{V}^\vee)$.

3 Abstract characterization of diagonal branching rules applied to generalized Verma modules for $so(n+1,1,\mathbb{R})$

The present section contains qualitative results on the diagonal branching rules for $so(n+1,1,\mathbb{R})$ applied to generalized Verma modules.

Let $n \in \mathbb{N}$ such that $n \geq 3$. Throughout the article $\mathfrak{g}$ denotes the real Lie algebra $so(n+1,1,\mathbb{R})$ of the connected and simply connected simple Lie group $G = SO_o(n+1,1,\mathbb{R})$. Let $\mathfrak{p}$ be its maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$, in the Dynkin diagrammatic notation for parabolic subalgebras given by omitting the first simple root of $\mathfrak{g}$. The Levi factor $\mathfrak{l}$

of $\mathfrak{p}$ is isomorphic to $so(n,\mathbb{R})\times\mathbb{R}$ and the commutative nilradical $\mathfrak{n}$ (resp. the opposite nilradical $\mathfrak{n}_-$) is isomorphic to $\mathbb{R}^n$. Let $diag : (\mathfrak{g},\mathfrak{p}) \hookrightarrow (\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$ denote the diagonal embedding.

The main task of the present article concerns the branching problem for the family of scalar generalized Verma $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$-modules induced from characters of the center of $\mathfrak{l}\oplus\mathfrak{l}$, with respect to $diag(\mathfrak{g},\mathfrak{p})$. An inducing character $\chi_{\lambda,\mu}$ on $\mathfrak{p}\oplus\mathfrak{p}$ is determined by two complex characters χ_μ,χ_λ on $\mathfrak{l}$:

$$\begin{aligned}\chi_{\lambda,\mu}\equiv(\chi_\lambda,\chi_\mu) \quad : \quad & \mathfrak{p}\oplus\mathfrak{p}\to\mathbb{C}\\ & (p_1,p_2)\mapsto\chi_\lambda(p_1)\cdot\chi_\mu(p_2), \end{aligned}\tag{6}$$

where the homomorphism (χ_λ,χ_μ) quotients through the semisimple subalgebra $[\mathfrak{p},\mathfrak{p}]\oplus[\mathfrak{p},\mathfrak{p}]$. The generalized Verma $(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$-module induced from character (χ_λ,χ_μ) $(\lambda,\mu\in\mathbb{C})$ is

$$\mathcal{M}(\mathfrak{g},\mathfrak{p},\mathbb{V}_{\lambda,\mu})\equiv\mathcal{M}_{\lambda,\mu}(\mathfrak{g},\mathfrak{p})=\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})}(\mathbb{C}_\lambda\otimes\mathbb{C}_\mu),\tag{7}$$

where $\mathbb{C}_\lambda\otimes\mathbb{C}_\mu$ is a 1-dimensional representation (χ_λ,χ_μ) of $\mathfrak{p}\oplus\mathfrak{p}$. As a vector space, $\mathcal{M}_{\lambda,\mu}(\mathfrak{g},\mathfrak{p})$ is isomorphic to the symmetric algebra $S^\star(\mathfrak{n}_-\oplus\mathfrak{n}_-)$, where $\mathfrak{n}_-\oplus\mathfrak{n}_-$ is the vector complement of $\mathfrak{p}\oplus\mathfrak{p}$ in $\mathfrak{g}\oplus\mathfrak{g}$.

A way to resolve this branching problem abstractly is based on character identities for the restriction of $\mathcal{M}_{\lambda,\mu}(\mathfrak{g},\mathfrak{p})$ to the diagonal subalgebra $diag(\mathfrak{g})$ with standard compatible parabolic subalgebra

$$diag(\mathfrak{p}):=diag(\mathfrak{g})\cap(\mathfrak{p}\oplus\mathfrak{p}),\ \mathfrak{p}=\mathfrak{l}\ltimes\mathfrak{n},\ diag(\mathfrak{p})=diag(\mathfrak{l})\ltimes diag(\mathfrak{n}).$$

For a fixed choice of positive simple roots of $\mathfrak{g}$ we denote by $\Lambda^+(\mathfrak{l}\oplus\mathfrak{l})$ the set of weights dominant for $\mathfrak{l}\oplus\mathfrak{l}$ and integral for $[\mathfrak{l},\mathfrak{l}]\oplus[\mathfrak{l},\mathfrak{l}]$. Let $\mathbb{V}_{\lambda,\mu}$ be a finite dimensional irreducible $\mathfrak{l}\oplus\mathfrak{l}$-module with highest weight $(\lambda,\mu)\in\Lambda^+(\mathfrak{l}\oplus\mathfrak{l})$, and likewise $\mathbb{V}_{\lambda'}$ be a finite dimensional representation of $diag(\mathfrak{l})$, $\lambda'\in\Lambda^+(diag(\mathfrak{l}))$. Given a vector space $\mathbb{V}$ we denote $S^\star(\mathbb{V})=\oplus_{l=0}^\infty S_l(\mathbb{V})$ the symmetric tensor algebra on $\mathbb{V}$. Let us extend the adjoint action of $diag(\mathfrak{l})$ on $(\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))$ to $S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)$. Notice that we have an isomorphism

$$(\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\simeq\mathfrak{n}_-$$

of $diag(\mathfrak{l})$-quotient modules. We set

$$\begin{aligned}& m\big(\lambda',(\lambda,\mu)\big):=\\ & Hom_{diag(\mathfrak{l})}\Big(\mathbb{V}_{\lambda'},\mathbb{V}_{\lambda,\mu}|_{diag(\mathfrak{l})}\otimes S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)\Big).\end{aligned}\tag{8}$$

Let us recall

Theorem 3.1 *([12], Theorem 3.9) Let $(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})$, $(\tilde{\mathfrak{g}}',\tilde{\mathfrak{p}}')$ be a compatible couple of simple Lie algebras and their parabolic subalgebras, and suppose $\tilde{\mathfrak{p}}$ is $\tilde{\mathfrak{g}}'$-compatible standard parabolic subalgebra of $\tilde{\mathfrak{g}}$, $(\lambda,\mu)\in\Lambda^+(\tilde{\mathfrak{l}}\oplus\tilde{\mathfrak{l}})$. Then*

1. *$m(\lambda',(\lambda,\mu))<\infty$ for all $\lambda'\in\Lambda^+(\tilde{\mathfrak{l}}')$.*

2. *In the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\tilde{\mathfrak{p}}'}$ there is $\tilde{\mathfrak{g}}'$-isomorphism*

$$\mathcal{M}_{\lambda,\mu}(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})|_{\tilde{\mathfrak{g}}'} \simeq \bigoplus_{\lambda'\in\Lambda^+(\tilde{\mathfrak{l}}')} m(\lambda',(\lambda,\mu))\mathcal{M}_{\lambda'}(\tilde{\mathfrak{g}}',\tilde{\mathfrak{p}}').$$

Applied to the case of our interest $(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}}) = (\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$, $(\tilde{\mathfrak{g}}',\tilde{\mathfrak{p}}') = diag(\mathfrak{g},\mathfrak{p}) = (diag(\mathfrak{g}),diag(\mathfrak{p}))$, we see that the polynomial ring $S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)$ decomposes as $diag(\mathfrak{l})$-module on irreducibles with higher multiplicities. In particular, each $diag(\mathfrak{l})$-module realized in homogeneity k polynomials also appears in polynomials of homogeneity $(k+2)$, $k\in\mathbb{N}$. As we already explained, we focus on the case of 1-dimensional inducing representations $\mathbb{V}_{\lambda,\mu}\simeq\mathbb{C}_\lambda\otimes\mathbb{C}_\mu$ as $(\mathfrak{l}\oplus\mathfrak{l})$-modules and $\mathbb{V}_{\lambda'}\simeq\mathbb{C}_\nu$ as $diag(\mathfrak{l})$-modules $(\lambda,\mu,\nu\in\mathbb{C})$. The multiplicity formula then implies that a nontrivial homomorphism in (8) occurs for each 1-dimensional $diag(\mathfrak{l})$-module in $S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)$, and it is a result in classical invariant theory (see [7],[14]) that for each even homogeneity there is just one 1-dimensional module. Because $\mathfrak{n}_-$ is as $(diag(\mathfrak{l})/[diag(\mathfrak{l}),diag(\mathfrak{l})])$-module isomorphic to the character $\mathbb{C}_{-1}$, the following relation holds true in the Grothendieck group of $\mathcal{O}^{(\mathfrak{p})}$ with $\mathfrak{p}\simeq diag(\mathfrak{p})$:

Corollary 3.2 *For $\mathfrak{g}\oplus\mathfrak{g} = so(n+1,1,\mathbb{R})\oplus so(n+1,1,\mathbb{R}), diag(\mathfrak{g})\simeq so(n+1,1,\mathbb{R})$ with standard maximal parabolic subalgebras $\mathfrak{p}\oplus\mathfrak{p}, diag(\mathfrak{p})$ given by omitting the first simple root in the corresponding Dynkin diagrams, $m(\nu,(\lambda,\mu)) = 1$ if and only if $\nu = \lambda+\mu-2j, j\in\mathbb{N}$ and $m(\nu,(\lambda,\mu)) = 0$ otherwise.*

Consequently, in the Grothendieck group of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$ holds

$$\mathcal{M}_{(\lambda,\mu)}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})|_{diag(\mathfrak{g})} \simeq \bigoplus_{j\in\mathbb{N}}\mathcal{M}_{\lambda+\mu-2j}(\mathfrak{g},\mathfrak{p}),$$

where $\nu = \lambda+\mu-2j$.

Although we work in one specific signature $(n+1,1)$, the results are easily extended to a real form of any signature.

4 The construction of singular vectors for diagonal branching rules applied to generalized Verma modules for $so(n+1,1,\mathbb{R})$

The rest of the article is devoted to the construction of singular vectors, whose abstract existence was concluded in the Section 3, Corollary 3.2, using the tool reviewed in Section 2. This can be regarded as a quantitative part of our diagonal branching problem.

4.1 Description of the representation

In this subsection we describe the representation of $\mathfrak{g}\oplus\mathfrak{g}$, acting upon the generalized Verma module

$$\mathcal{M}_{\lambda,\mu}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})=\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})}(\mathbb{C}_{\lambda,\mu}),\ \mathbb{C}_{\lambda,\mu}=\mathbb{C}_\lambda\otimes\mathbb{C}_\mu,$$

in its Fourier image, i.e. apply the framework for the F-method explained in Section 2 to

$$\mathcal{M}_{\lambda,\mu}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})\simeq\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})\otimes\mathcal{M}_\mu(\mathfrak{g},\mathfrak{p}). \tag{9}$$

The first goal is to describe the action of elements in the nilradical $diag(\mathfrak{n})$ of $diag(\mathfrak{p})$ in terms of differential operators acting on the Fourier image of $\mathcal{M}_{\lambda,\mu}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$. This can be derived from the explicit form of the action on the induced representation realized in the non-compact picture, and it follows from (9) that the problem can be reduced to the question on each component of the tensor product separately. Let us consider the complex representation π_λ, $\lambda\in\mathbb{C}$, of $G=SO_o(n+1,1,\mathbb{R})$, induced from the character $p\mapsto a^\lambda$, $p\in P$, acting on the one dimensional representation space $\mathbb{C}_\lambda\simeq\mathbb{C}$. Here $a\in A=\mathbb{R}^\star$ is the abelian subgroup in the Langlands decomposition $P=MAN$, $M=SO(n)$, $N=\mathbb{R}^n$. In other words, π_λ acts by left regular representation on $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$.

Let x_j be the coordinates with respect to the standard basis on $\mathfrak{n}_-$, and ξ_j, $j=1,\dots,n$ the coordinates on the Fourier transform of $\mathfrak{n}_-$. We consider the family of differential operators

$$Q_j(\lambda)=-\frac{1}{2}|x|^2\partial_j+x_j(-\lambda+\sum_k x_k\partial_k),\ j=1,\dots,n, \tag{10}$$

$$P_j^\xi(\lambda)=i\left(\frac{1}{2}\xi_j\triangle^\xi+(\lambda-\mathbb{E}^\xi)\partial_{\xi_j}\right),\ j=1,\dots,n, \tag{11}$$

where $|x|^2=x_1^2+\cdots+x_n^2$,

$$\triangle^\xi=\partial_{\xi_1}^2+\cdots+\partial_{\xi_n}^2$$

is the Laplace operator of positive signature, $\partial_j=\frac{\partial}{\partial x_j}$ and $\mathbb{E}^\xi=\sum_k\xi_k\partial_{\xi_k}$ is the Euler homogeneity operator ($i\in\mathbb{C}$ the complex unit.) The following result is a routine computation:

Lemma 4.1 *([10]) Let us denote by E_j the standard basis elements of $\mathfrak{n}$, $j=1,\dots,n$. Then $E_j\in\mathfrak{n}$ are acting on $C^\infty(\mathfrak{n}_-,\mathbb{C}_{-\lambda})$ by*

$$d\tilde{\pi}_\lambda(E_j)(s\otimes v)=Q_j(\lambda)(s)\otimes v,\ s\in C^\infty(\mathfrak{n}_-,\mathbb{C}),\ v\in\mathbb{C}_{-\lambda}, \tag{12}$$

and the action of $(d\tilde{\pi})^F_\lambda$ on $\mathrm{Pol}[\xi_1,\dots,\xi_n]\otimes\mathbb{C}_\lambda^\vee$ is given by

$$(d\tilde{\pi})^F_\lambda(E_j)(f\otimes v)=P_j^\xi(\lambda)(f)\otimes v,\ f\in\mathrm{Pol}[\xi_1,\dots,\xi_n],\ v\in\mathbb{C}_{-\lambda^\vee}. \tag{13}$$

As for the action of remaining basis elements of $\mathfrak{g}$ in the Fourier image of the induced representation, the action of $\mathfrak{n}_-$ is given by multiplication by coordinate functions, the standard basis elements of the simple part of the Levi factor $[\mathfrak{l},\mathfrak{l}]=so(n)$ act by differential operators

$$M_{ij}^\xi=(\xi_j\partial_{\xi_i}-\xi_i\partial_{\xi_j}),\ i,j=1,\dots,n$$

and the basis element of the Lie algebra of A acts as the homogeneity operator, $\mathbb{E}^\xi = \sum_{i=1}^n \xi_i \partial_{\xi_i}$.

In the Fourier image of the tensor product of two induced representations in the non-compact realization on $\mathfrak{n}_- \oplus \mathfrak{n}_-$ with coordinates ξ_i resp. ν_i on the first resp. second copy of $\mathfrak{n}_-$ in $\mathfrak{n}_- \oplus \mathfrak{n}_-$, the generators of the diagonal subalgebra $diag(\mathfrak{g})$ act on the representation $\mathrm{Ind}_{P\times P}^{G\times G}(\mathbb{C}_{\lambda,\mu})$ induced from (χ_λ, χ_μ) as

1. Multiplication by
$$(\xi_j \otimes 1) + (1 \otimes \nu_j),\ j = 1, \dots, n \tag{14}$$
for the elements of $diag(\mathfrak{n}_-)$,

2. First order differential operators with linear coefficients
$$\begin{aligned} M_{ij}^{\xi,\nu} &= (M_{ij}^\xi \otimes 1) + (1 \otimes M_{ij}^\nu) \\ &= (\xi_j \partial_{\xi_i} - \xi_i \partial_{\xi_j}) \otimes 1 + 1 \otimes (\nu_j \partial_{\nu_i} - \nu_i \partial_{\nu_j}), \end{aligned} \tag{15}$$
$i, j = 1, \dots, n$ for the elements of the simple subalgebra of $diag(\mathfrak{l})$ and
$$\mathbb{E}^\xi \otimes 1 + 1 \otimes \mathbb{E}^\nu = \sum_{i=1}^n (\xi_i \partial_{\xi_i} \otimes 1 + 1 \otimes \nu_i \partial_{\nu_i}), \tag{16}$$
for the generator of $diag(\mathfrak{l})/[diag(\mathfrak{l}), diag(\mathfrak{l})]$,

3. Second order differential operators with linear coefficients
$$\begin{aligned} P_j^{\xi,\nu}(\lambda,\mu) &= (P_j^\xi(\lambda) \otimes 1) + (1 \otimes P_j^\nu(\mu)) \\ &= i(\frac{1}{2}\xi_j \triangle^\xi + (\lambda - \mathbb{E}^\xi)\partial_{\xi_j}) \otimes 1 \\ &\quad + i1 \otimes (\frac{1}{2}\nu_j \triangle^\nu + (\mu - \mathbb{E}^\nu)\partial_{\nu_j}), \end{aligned} \tag{17}$$
$j = 1, \dots, n$ for the elements $diag(\mathfrak{n})$.

This completes the description of the first part of abstract procedure outlined in Section 2 in the case of the diagonal branching problem of our interest.

4.2 Reduction to a hypergeometric differential equation in two variables

It follows from the discussion in Sections 2, 3 that $diag(\mathfrak{l})$-modules inducing singular vectors for the diagonal branching rules are one dimensional. This means that they are annihilated by $diag(\mathfrak{l}^s) = diag([\mathfrak{l}, \mathfrak{l}]) \simeq so(n, \mathbb{R})$, the simple part of the diagonal Levi factor $diag(\mathfrak{l}) \simeq so(n, \mathbb{R}) \times \mathbb{R}$. It follows that the singular vectors are invariants of $diag(\mathfrak{l})$ acting diagonally on the algebra of polynomials on $\mathfrak{n}_- \oplus \mathfrak{n}_-$ as a $\mathfrak{l} \oplus \mathfrak{l}$-module. The following result is a special case of the first fundamental theorem in classical invariant theory, see e.g. [7], [14].

Lemma 4.2 *Let $(V, <,>)$ be a finite dimensional real vector space with bilinear form $<,>$ and $SO(V)$ the Lie group of automorphisms of $(V, <,>)$. Then the subalgebra of $SO(V)$-invariants in the complex polynomial algebra $Pol[V \oplus V]$ ($SO(V)$ acting diagonally on $V \oplus V$) is polynomial algebra generated by $< \xi, \xi >$, $< \xi, \nu >$ and $< \nu, \nu >$. Here we use the convention that ξ is a vector in the first component V of $V \oplus V$ and ν in the second summand.*

In our case, the complex polynomial algebra is $Pol[\xi_1, \dots, \xi_n, \nu_1, \dots, \nu_n]$ and we use the notation $Pol[r, s, t]$ for the (complex) subalgebra of invariants:

$$
\begin{aligned}
r &:=< \xi, \nu >= \sum_{i=1}^{n} \xi_i \nu_i, \\
s &:=< \xi, \xi >= \sum_{i=1}^{n} \xi_i \xi_i, \\
t &:=< \nu, \nu >= \sum_{i=1}^{n} \nu_i \nu_i.
\end{aligned} \tag{18}
$$

The task of the present subsection is to rewrite the operators $P_j^{\xi,\nu}(\lambda, \mu)$ in the variables r, s, t, i.e. we reduce the action of $P_j^{\xi,\nu}(\lambda, \mu)$ from the polynomial ring to the ring of $diag(\mathfrak{n}_-)$-invariants on $\mathfrak{n}_- \oplus \mathfrak{n}_-$.

We compute

$$
\partial_{\nu_i} r = \xi_i,\ \partial_{\xi_i} r = \nu_i, \partial_{\nu_i} s = 0,\ \partial_{\xi_i} s = 2\xi_i, \partial_{\nu_i} t = 2\nu_i,\ \partial_{\xi_i} t = 0, \tag{19}
$$

and

$$
\partial_{\xi_i} = \nu_i \partial_r + 2\xi_i \partial_s, \triangle^{\xi} = t\partial_r^2 + 4r\partial_r\partial_s + 2n\partial_s + 4s\partial_s^2,\ i = 1, \dots, n. \tag{20}
$$

Note that analogous formulas for ∂_{ν_i}, $\triangle^{\nu}$ can be obtained from those for ξ by applying the involution

$$
\xi_i \longleftrightarrow \nu_i,\ s \longleftrightarrow t,\ r \longleftrightarrow r. \tag{21}
$$

We also have for all $i = 1, \dots, n$

$$
\mathbb{E}^{\xi} \partial_{\xi_i} = \nu_i (\mathbb{E}^r + 2\mathbb{E}^s)\partial_r + \xi_i(2\mathbb{E}^r + 4\mathbb{E}^s + 2)\partial_s, \tag{22}
$$

and so taking all together we arrive at the operators

$$
\begin{aligned}
P_i^{r,s,t}(\lambda, \mu) &= \xi_i(\frac{1}{2}t\partial_r^2 + (n + 2\lambda - 2 - 2\mathbb{E}^s)\partial_s - (\mathbb{E}^r + 2\mathbb{E}^t - \mu)\partial_r) \\
&+ \nu_i(\frac{1}{2}s\partial_r^2 + (n + 2\mu - 2 - 2\mathbb{E}^t)\partial_t - (\mathbb{E}^r + 2\mathbb{E}^s - \lambda)\partial_r)
\end{aligned} \tag{23}
$$

acting on complex polynomial algebra $Pol[r,s,t]$, $i=1,\dots,n$. A consequence of the system of equations ($i=1,\dots,n$) is

$$
\begin{aligned}
& P^{r,s,t}_{\xi}(\lambda,\mu) := \sum_{i=1}^{n}\xi_i P^{r,s,t}_i(\lambda,\mu) = s(\frac{1}{2}t\partial_r^2 + (n+2\lambda-2-2\mathbb{E}^s)\partial_s \\
& -(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r) + r(\frac{1}{2}s\partial_r^2 + (n+2\mu-2-2\mathbb{E}^t)\partial_t - (\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r), \\
& P^{r,s,t}_{\nu}(\lambda,\mu) := \sum_{i=1}^{n}\nu_i P^{r,s,t}_i(\lambda,\mu) = r(\frac{1}{2}t\partial_r^2 + (n+2\lambda-2-2\mathbb{E}^s)\partial_s \\
& -(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r) + t(\frac{1}{2}s\partial_r^2 + (n+2\mu-2-2\mathbb{E}^t)\partial_t - (\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r).
\end{aligned}
\tag{24}
$$

Notice that the second equation follows from the first one by the action of involution

$$\lambda \longleftrightarrow \mu,\ s \longleftrightarrow t,\ r \longleftrightarrow r.$$

In what follows we construct a set of homogeneous polynomial solutions of $P^{r,s,t}_{\xi}(\lambda,\mu)$, $P^{r,s,t}_{\nu}(\lambda,\mu)$ solving the system $\{P^{r,s,t}_i(\lambda,\mu)\}_i$, $i=1,\dots,n$. The uniqueness of the solution for the generic values of the inducing parameters implies the unique solution of the former system of PDEs.

Notice that (24) is the system of differential operators preserving the space of homogeneous polynomials in the variables r,s,t, i.e. $P^{r,s,t}_{\xi}(\lambda,\mu)$, $P^{r,s,t}_{\nu}(\lambda,\mu)$ commute with $\mathbb{E}^{r,s,t} := \mathbb{E}^r + \mathbb{E}^s + \mathbb{E}^t$.

Let $p=p(r,s,t)$ be a homogeneous polynomial of degree N, $deg(p) = N$, and write

$$
\begin{aligned}
& p = r^N p(\frac{s}{r},\frac{t}{r}) = r^N\tilde{p}(u,v),\ u := \frac{s}{r}, v := \frac{t}{r}, \\
& \tilde{p}(u,v) = \sum_{i,j|0\leq|i+j|\leq N} A_{i,j}u^i v^j,
\end{aligned}
\tag{25}
$$

where $\tilde{p}$ is the polynomial of degree N. We have

$$\partial_s = \frac{1}{r}\partial_u + \frac{1}{u}\partial_r,\ \partial_t = \frac{1}{r}\partial_v + \frac{1}{v}\partial_r$$

and so the summands in (24) transform as

$$
\begin{aligned}
& \frac{1}{2}st\partial_r^2 = \frac{1}{2}N(N-1)uv, \\
& s(n+2\lambda-2-2\mathbb{E}^s)\partial_s = (n+2\lambda-2\mathbb{E}^u-2N)(\mathbb{E}^u+N), \\
& -s(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r = -N(3N-3+2\mathbb{E}^v-\mu)u, \\
& \frac{1}{2}rs\partial_r^2 = \frac{1}{2}N(N-1)u, \\
& r(n+2\mu-2-2\mathbb{E}^t)\partial_t = (n+2\mu-2\mathbb{E}^v-2N)(\partial_v+\frac{N}{v}), \\
& -r(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r = -N(2\mathbb{E}^u-\lambda+3N-3)
\end{aligned}
\tag{26}
$$

when acting on $r^N\tilde{p}(u,v)$. Taken all together, we get

Lemma 4.3 *The substitution (25) transforms the former system of PDE's (24) to a hypergeometric differential operator*

$$\begin{aligned} P_\xi^{u,v}(\lambda,\mu) &= \frac{1}{2}N(N-1)uv + (n+2\lambda-2\mathbb{E}^u-2N)(\mathbb{E}^u+N) \\ &\quad -N(3N-3+2\mathbb{E}^v-\mu)u + \frac{1}{2}N(N-1)u \\ &\quad +(n+2\mu-2\mathbb{E}^v-2N)(\partial_v+\frac{N}{v}) - N(2\mathbb{E}^u-\lambda+3N-3), \end{aligned} \tag{27}$$

fulfilling

$$P_\nu^{u,v}(\lambda,\mu) = P_\xi^{v,u}(\mu,\lambda).$$

In the next subsection we find, for generic values of the inducing parameters λ,μ, a unique solution for this hypergeometric equation for a given homogeneity.

4.3 Solution of the hypergeometric differential equation in two variables

We start with a couple of simple examples.

Example 4.4 *Let us consider the polynomial of homogeneity one,*

$$p(r,s,t) = Ar + Bs + Ct, \;\; A,B,C \in \mathbb{C}.$$

The application of $P_i^{r,s,t}(\lambda,\mu)$ yields

$$P_i^{r,s,t}(\lambda,\mu)(Ar+Bs+Ct) = \xi_i(B(n+2\lambda-2)+A\mu) + \nu_i(C(n+2\mu-2)+A\lambda) \tag{28}$$

for all $i=1,\dots,n$. When A is normalized to be equal to 1, we get

$$C = -\frac{\lambda}{n+2\mu-2}, \; B = -\frac{\mu}{n+2\lambda-2}.$$

The unique homogeneous (resp. non-homogeneous) solution of $P_i^{r,s,t}(\lambda,\mu)$ for all $i=1,\dots,n$ is then

$$p(r,s,t) = (n+2\lambda-2)(n+2\mu-2)r - \mu(n+2\mu-2)s - \lambda(n+2\lambda-2)t. \tag{29}$$

Example 4.5 *Let*

$$p(r,s,t) = Ar^2 + Bs^2 + Ct^2 + Drs + Est + Frt, \;\; A,B,C,D,E,F \in \mathbb{C}$$

be a general polynomial of homogeneity two. We have

$$\begin{aligned} &P_i^{r,s,t}(\lambda,\mu)p(r,s,t) = \xi_i[r(D(n+2\lambda-2)+A2(\mu-1)) + s(B2(n+2\lambda-4)+D\mu) \\ &+t(A+E(n+2\lambda-2)+F(\mu-2))], \\ &+\nu_i[r(F(n+2\mu-2)+A2(\lambda-1)) + s(A+E(n+2\mu-2)+D(\lambda-2)) \\ &+t(C2(n+2\mu-4)+F\lambda)], \end{aligned} \tag{30}$$

for all $i=1,\dots,n$. The equations

$$\sum_i \xi_i P_{\xi_i}^{r,s,t}(\lambda,\mu) = 0, \;\; \sum_i \nu_i P_{\nu_i}^{r,s,t}(\lambda,\mu) = 0$$

are equivalent to two systems of linear equations:

$$
\begin{aligned}
&D(n+2\lambda-2)+A2(\mu-1)+A+E(n+2\mu-2)+D(\lambda-2)=0,\\
&F(n+2\mu-2)+A2(\lambda-1)=0,\\
&B2(n+2\mu-4)+D\mu=0,\\
&A+E(n+2\lambda-2)+F(\mu-2)=0,\\
&C2(n+2\mu-4)+F\lambda)=0, \qquad (31)
\end{aligned}
$$

resp.

$$
\begin{aligned}
&D(n+2\lambda-2)+A2(\mu-1)=0,\\
&B2(n+2\mu-4)+D\mu=0,\\
&A+E(n+2\lambda-2)+F(\mu-2)+F(n+2\mu-2)+A2(\lambda-1)=0,\\
&A+E(n+2\mu-2)+D(\lambda-2)=0,\\
&C2(n+2\mu-4)+F\lambda=0. \qquad (32)
\end{aligned}
$$

Both systems are equivalent under the involution

$$A\longleftrightarrow A,\ E\longleftrightarrow E,\ D\longleftrightarrow F,\ B\longleftrightarrow C,\ \lambda\longleftrightarrow\mu$$

and its unique solution invariant under this involution is

$$
\begin{aligned}
&A=1,\ F=\frac{-2(\lambda-1)}{n+2\mu-2},\ C=\frac{\lambda(\lambda-1)}{(n+2\mu-2)(n+2\mu-4)},\\
&E=2\frac{(\lambda-2)(\mu-2)-(1+\frac{n}{2})}{(n+2\mu-2)(n+2\lambda-2)},\ D=\frac{-2(\mu-1)}{(n+2\lambda-2)},\\
&B=\frac{\mu(\mu-1)}{(n+2\lambda-2)(n+2\lambda-4)}. \qquad (33)
\end{aligned}
$$

The vector

$$
\begin{aligned}
p(r,s,t) \quad = \quad & (n+2\lambda-2)(n+2\lambda-4)(n+2\mu-2)(n+2\mu-4)r^2\\
&+\mu(\mu-1)(n+2\mu-2)(n+2\mu-4)s^2\\
&+\lambda(\lambda-1)(n+2\lambda-2)(n+2\lambda-4)t^2\\
&-2(\mu-1)(n+2\lambda-4)(n+2\mu-2)(n+2\mu-4)rs\\
&+2((\lambda-2)(\mu-2)-(1+\frac{n}{2}))(n+2\lambda-4)(n+2\mu-4)st\\
&-2(\lambda-1)(n+2\lambda-2)(n+2\lambda-4)(n+2\mu-4)rt. \qquad (34)
\end{aligned}
$$

is then the unique solution of $P_i^{r,s,t}(\lambda,\mu)$ *of homogeneity two.*

We now return back to the situation of a general homogeneity. The dehomogenesation $(r,s,t,)\to(r,u,v)$ is governed by coordinate change

$$u:=\frac{s}{r},\ v:=\frac{t}{r},\ r:=r, \qquad (35)$$

and so

$$
\begin{aligned}
&\partial_s\to\frac{1}{r}\partial_u,\ \partial_t\to\frac{1}{r}\partial_v,\ \partial_r\to-\frac{1}{r}u\partial_u-\frac{1}{r}v\partial_v+\partial_r\,,\\
&\mathbb{E}^s\to\mathbb{E}^u,\ \mathbb{E}^t\to\mathbb{E}^v,\ \mathbb{E}^r\to-\mathbb{E}^u-\mathbb{E}^v+\mathbb{E}^r\,. \qquad (36)
\end{aligned}
$$

The terms in $P_\xi^{r,s,t}(\lambda,\mu)$ transform into

$$\begin{aligned}
\frac{1}{2}st\partial_r^2 &\to \frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r+1)(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r),\\
-r(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r &\to (\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r)(\mathbb{E}^u-\mathbb{E}^v+\mathbb{E}^r-\lambda-1),\\
r(n+2\mu-2-2\mathbb{E}^t)\partial_t &\to (n+2\mu-2-2\mathbb{E}^v)\partial_v,\\
\frac{1}{2}rs\partial_r^2 &\to \frac{1}{2}u(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r+1)(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r),\\
-s(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r &\to u(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r)(-\mathbb{E}^u+\mathbb{E}^v+\mathbb{E}^r-\mu-1),\\
s(n+2\lambda-2-2\mathbb{E}^s)\partial_s &\to (n+2\lambda-2\mathbb{E}^u)\mathbb{E}_u,
\end{aligned}\tag{37}$$

and when acting on a polynomial of homogeneity N, $p(r,s,t)=r^N\tilde{p}(u,v)$ for a polynomial $\tilde{p}(u,v)$ of degree N in u,v, $\mathbb{E}^r=N$ and we get

$$\begin{aligned}
P_\xi^{u,v}(\lambda,\mu) = &\frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-N+1)(\mathbb{E}^u+\mathbb{E}^v-N)\\
&-(\mathbb{E}^u)^2+\mathbb{E}^u(n+\lambda-1)+(\mathbb{E}^v-N)(-\mathbb{E}^v+N-\lambda-1)\\
&+(n+2\mu-2-2\mathbb{E}^v)\partial_v\\
&+\frac{1}{2}u(\mathbb{E}^u+\mathbb{E}^v-N)(-\mathbb{E}^u+3\mathbb{E}^v+N-2\mu-1).
\end{aligned}\tag{38}$$

Similarly, one gets

$$\begin{aligned}
P_\nu^{u,v}(\lambda,\mu) = &\frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-N+1)(\mathbb{E}^u+\mathbb{E}^v-N)\\
&-(\mathbb{E}^v)^2+\mathbb{E}^v(n+\mu-1)+(\mathbb{E}^u-N)(-\mathbb{E}^u+N-\mu-1)\\
&+(n+2\lambda-2-2\mathbb{E}^u)\partial_u\\
&+\frac{1}{2}v(\mathbb{E}^v+\mathbb{E}^u-N)(-\mathbb{E}^v+3\mathbb{E}^u+N-2\lambda-1).
\end{aligned}\tag{39}$$

Let us denote $A_{i,j}(\lambda,\mu)$ the coefficient by monomial u^iv^j in the polynomial $\tilde{p}(u,v)$. The assumption $A_{i,j}(\lambda,\mu)=A_{j,i}(\mu,\lambda)$, combined with the symmetry between $P_\xi^{u,v}(\lambda,\mu)$ and $P_\nu^{u,v}(\lambda,\mu)$, allows to restrict to the action of $P_\xi^{u,v}(\lambda,\mu)$ on a polynomial of degree N of the form

$$\tilde{p}(u,v)=\sum_{i,j|i\le N,\, j\le N} A_{i,j}(\lambda,\mu)u^iv^j,\ \ A_{i,j}(\lambda,\mu)=A_{j,i}(\mu,\lambda),\tag{40}$$

thereby converting the differential equation (38) into the four-term functional relation

$$\begin{aligned}
&\frac{1}{2}(i+j-N-1)(i+j-N-2)A_{i-1,j-1}(\lambda,\mu)\\
&+(-i^2+i(n+\lambda-1)+(j-N)(-j+N-\lambda-1)A_{i,j}(\lambda,\mu)\\
&+(j+1)(n+2\mu-2-2j)A_{i,j+1}(\lambda,\mu)\\
&+\frac{1}{2}(i+j-N-1)(-i+3j+N-2\mu)A_{i-1,j}(\lambda,\mu)=0
\end{aligned}\tag{41}$$

for $i,j=1,\dots,N$ and $j\ge i$, which recursively computes $A_{i,j+1}(\lambda,\mu)$ in terms of $A_{i-1,j-1}(\lambda,\mu), A_{i-1,j}(\lambda,\mu)$ and $A_{i,j}(\lambda,\mu)$.

There is still one question we have not mentioned yet, concerning the normalization of $A_{i,j}(\lambda,\mu)$. A singular vector can be normalized by

multiplication by common denominator resulting in the coefficients valued in $Pol[\lambda,\mu]$ rather than the field $\mathbb{C}(\lambda,\mu)$. As we shall prove in the next Theorem, a consequence of (41) is the uniqueness of its solution in the range $\lambda,\mu\notin\{m-\frac{n}{2}|m\in\mathbb{N}\}$. We observe that the uniqueness of solution fails for $\lambda,\mu\in\{m-\frac{n}{2}|m\in\mathbb{N}\}$, which indicates the appearance of a non-trivial composition structure in the branching problem for generalized Verma modules.

In the following Theorem we construct a set of singular vectors, which will be the representatives realizing abstract character formulas of the diagonal branching problem in Corollary 3.2.

Theorem 4.6 *Let $\lambda,\mu\in\mathbb{C}\setminus\{m-\frac{n}{2}|m\in\mathbb{N}\}$, $N\in\mathbb{N}$, and $(x)_l=x(x+1)\dots(x+l-1)$, $l\in\mathbb{N}$ be the Pochhamer symbol for $x\in\mathbb{C}$. The four-term functional equation (41) for the set $\{A_{i,j}(\lambda,\mu)\}_{i,j\in\{1,\dots,N\}}$ fulfilling*

$$A_{j,i}(\lambda,\mu)=A_{i,j}(\mu,\lambda),\ j\geq i,$$

has a unique nontrivial solution

$$\begin{aligned}&A_{i,j}(\lambda,\mu)=\\&\frac{\Gamma(i+j-N)\Gamma(1-\frac{n}{2}-\mu)\Gamma(1-i+j-N+\lambda)\Gamma(\lambda+\frac{n}{2}-i)}{2^{i+j}(-)^{i+j}i!j!\,\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)\Gamma(\lambda+\frac{n}{2})}\cdot\\&\sum_{k=0}^{i}(-)^k\binom{i}{k}(j-i+1+k)_{i-k}(\lambda+\frac{n}{2}-i)_{i-k}(\mu-N+1)_k(\lambda-N+1-k)_k.\end{aligned}\tag{42}$$

Out of the range $\lambda,\mu\in\mathbb{C}\setminus\{m-\frac{n}{2}|m\in\mathbb{N}\}$, (42) is still a solution, but not necessarily unique.

Proof:

Let us first discuss the uniqueness of the solution. The knowledge of $A_{i,j}(\lambda,\mu)$ for $i+j\leq k_0$ allows to compute the coefficient $A_{i,j+1}(\lambda,\mu)$ with $i+j=k_0+1$ from the recursive functional equation, because of assumption $\lambda,\mu\notin\{m-\frac{n}{2}|m\in\mathbb{N}\}$. The symmetry condition for $A_{i,j}(\lambda,\mu)$ gives $A_{j,i}(\mu,\lambda)=A_{i,j}(\lambda,\mu)$ and the induction proceeds by passing to the computation of $A_{i,j+2}(\lambda,\mu)$. Note that all coefficients are proportional to $A_{0,0}(\lambda,\mu)$ and its choice affects their explicit form.

The proof of the explicit form for $A_{i,j}(\lambda,\mu)$ is based on the verification of the recursion functional equation (41). To prove that the left hand side of (41) is trivial is equivalent to the following check: up to a product of linear factors coming from Γ-functions, the left hand side is the sum of four polynomials in λ,μ. A simple criterion for the triviality of a polynomial of degree d we use is that it has d roots (counted with multiplicity) and the leading monomial in a corresponding variable has coefficient zero.

It is straightforward but tedious to check that the left hand side of (41) has, as a polynomial in λ, the roots $\lambda=k-\frac{n}{2}$ for $k=1,\dots,i$ and its leading coefficient is zero. Let us first consider $\lambda=i-\frac{n}{2}$, so get after substitution

$$\begin{aligned}&A_{i,j}(i-\frac{n}{2},\mu)=\frac{(-)^j(i+j-N-1)\dots(-N)}{2^{i+j}i!j!}\cdot\\&\frac{(j-N-\frac{n}{2})\dots(i-N-\frac{n}{2}+1)(1-\frac{n}{2}-N)_i(\mu-N+1)_i}{(j-\frac{n}{2}-\mu)\dots(1-\frac{n}{2}-\mu)(\lambda+\frac{n}{2}-1)\dots(\lambda+\frac{n}{2}-i)}\end{aligned}\tag{43}$$

and

$$A_{i,j+1}(i-\frac{n}{2},\mu)=A_{i,j}(i-\frac{n}{2},\mu)\cdot\frac{(-)(i+j-N)(j-N-\frac{n}{2}+1)}{2(j+1)(j-\frac{n}{2}-\mu+1)}. \quad (44)$$

Taken together, there remain just two contributions on the left hand side of (41) given by $A_{i,j}(i-\frac{n}{2},\mu)$, $A_{i,j+1}(i-\frac{n}{2},\mu)$. Up to a common rational factor, their sum is proportional to

$$\begin{aligned} & i(\frac{n}{2}-1)+(j-N)(-j+N-i+\frac{n}{2}-1)+ \\ & (j+1)(n+2\mu-2-2j)(-)\frac{(i+j-N)(j-N-\frac{n}{2}+1)}{2(j+1)(j-\frac{n}{2}-\mu+1)}=0, \end{aligned}$$

which proves the claim. The proof of triviality of the left hand side at special values $\lambda=i-1-\frac{n}{2},\dots,1-\frac{n}{2}$ is completely analogous.

Note that there are some other equally convenient choices for λ,μ allowing the triviality check for (41), for example based on the choice $\lambda=k+N-1$, $k=1,\dots,i$ or $\mu=N-k$, $k=1,\dots,i$.

The remaining task is to find the leading coefficient on the left hand side of (41) as a polynomial in λ. Because

$$\begin{aligned} (\lambda+\frac{n}{2}-i)_{i-k} & \overset{\lambda\to\infty}{\sim} \lambda^{i-k}, \\ (\lambda-N+1-k)_k & \overset{\lambda\to\infty}{\sim} \lambda^{k}, \end{aligned} \quad (45)$$

the polynomial is of degree $\lambda^{j-i}\frac{\lambda^i}{\lambda^i}=\lambda^{j-i}$, $j\geq i$. The leading coefficient of $A_{i,j}(\lambda,\mu)$ is

$$\begin{aligned} \lim_{\lambda\to\infty}\frac{A_{i,j}(\lambda,\mu)}{\lambda^{j-i}} &= (\sum_{k=0}^{i}(-)^k\binom{i}{k}(j-i+1+k)_{i-k}(\mu-N+1)_k)\cdot \\ & \frac{(-)^{i+j}(i+j-N-1)\dots(-N)}{2^{i+j}i!j!\,(j-\frac{n}{2}-\mu)\dots(1-\frac{n}{2}-\mu)}. \end{aligned} \quad (46)$$

There are three contributions to (41):

$$\begin{aligned} & (N-j+i)\lim_{\lambda\to\infty}\frac{A_{i,j}(\lambda,\mu)}{\lambda^{j-i}}, \\ & (j+1)(n+2\mu-2-2j)\lim_{\lambda\to\infty}\frac{A_{i,j+1}(\lambda,\mu)}{\lambda^{j+1-i}}, \\ & \frac{1}{2}(i+j-N-1)(-i+3j+N-2\mu)\lim_{\lambda\to\infty}\frac{A_{i-1,j}(\lambda,\mu)}{\lambda^{j+1-i}}, \end{aligned} \quad (47)$$

whose sum is a polynomial in μ multiplied by common product of linear polynomial. In order to prove triviality of this polynomial, it suffices as in the first part of the proof to find sufficient amount of its roots and to prove the triviality of its leading coefficient. For example in the case $\mu=N-1$, we get from (47) that the coefficients of this polynomial are proportional to the sum

$$(N-j+i)+\frac{(j+1)(i+j-N)}{(j-i+1)}-\frac{i(-i+3j+N-2(N-1))}{(j-i+1)},$$

which equals to zero. The verification of the required property for $\mu = N-k$, $k=2,\dots,i$ is completely analogous. This completes the proof.

$\square$

This completes the description of the set $Sol(\mathfrak{g}\oplus\mathfrak{g}, diag(\mathfrak{g}), \mathbb{C}_{\lambda,\mu})$ characterizing solution space of a diagonal branching problem for $so(n+1,1,\mathbb{R})$, (4).

Remark 4.7 *It is an interesting observation that the four term functional equation (41) for $A_{i,j}(\lambda,\mu)$ can be simplified using hypergeometric functions ${}_3F_2$:*

$$
{}_3F_2(a_1,a_2,a_3;b_1,b_2;z) := \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(a_3)_m}{(b_1)_m(b_2)_m}\frac{z^m}{m!},
$$

where $a_1,a_2,a_3,b_1,b_2 \in \mathbb{C}$ and $(x)_m = x(x+1)\dots(x+m-1)$. In particular, it can be converted into four term functional equation

$$
\begin{aligned}
&\frac{(n+2\lambda)\Gamma(i+j-N)\Gamma(-\frac{n}{2}-\lambda)\Gamma(1-i+j-N+\lambda)\Gamma(1-\frac{n}{2}-\mu)}{2^{i+j}(-)^{i+j}\Gamma(1+i)\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)}\cdot\\
&(i(-2j+n+2\mu){}_3F_2(1-i,N-\lambda,1-N+\mu;1-i+j,1-\frac{n}{2}-\lambda;1)+\\
&i(-1+i-j+N-\lambda)(i-3j-N+2\mu)\cdot\\
&\cdot{}_3F_2(1-i,N-\lambda,1-N+\mu;2-i+j,1-\frac{n}{2}-\lambda;1)+\\
&(i^2-i(-1+n+\lambda)+(j-N)(1+j-N+\lambda))\cdot\\
&\cdot{}_3F_2(-i,N-\lambda,1-N+\mu;1-i+j,1-\frac{n}{2}-\lambda;1)+\\
&(1+j)(i+j-N)(-1+i-j+N-\lambda)\cdot\\
&\cdot{}_3F_2(-i,N-\lambda,1-N+\mu;2-i+j,1-\frac{n}{2}-\lambda;1))=0.
\end{aligned}
\tag{48}
$$

This functional equation, whose knowledge would clearly simplify the formulation of the proof of the last Theorem, is advanced to be found in any standard textbook on special function theory of several variables (see e.g., [6], [1]), and does not seem to be accessible in the literature.

In fact, a large part of the monograph [1] is devoted to evaluations of generalized hypergeometric functions ${}_{p+1}F_p$ at $z=1$, at least for reasonably small values of $p\in\mathbb{N}$. However, the Saalschutz's theorems of the form

$$
\begin{aligned}
&\sum_{r=0}^{n}\frac{(\frac{1}{2}a)_r(\frac{1}{2}+\frac{1}{2}a-b)_r(-4)^r(a+2r)_{n-r}}{r!(n-r)!(1+a-b)_r} =\\
&\frac{(a)_n}{n!}{}_3F_2(\frac{1}{2}+\frac{1}{2}a-b,a+n,-n;1+a-b,\frac{1}{2}+\frac{1}{2}a;1)
\end{aligned}
\tag{49}
$$

are too special and restrictive to be of direct use for our needs.

Example 4.8 *As an example, we have*

$$
\begin{aligned}
A_{1,j}(\lambda,\mu) \quad = \quad & \frac{\Gamma(1+j-N)\Gamma(j-N+\lambda)\Gamma(1-\frac{n}{2}-\mu)}{2^{j+1}(-)^{j+1}\Gamma(1+j)\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)}\cdot\\
&\cdot\frac{(j(-2+n+2\lambda)+2(N-\lambda)(1-N+\mu))}{(n+2\lambda-2)}
\end{aligned}
\tag{50}
$$

for all $j \in \{1, \ldots, N\}$.

Let us also remark that for special values $\lambda, \mu \in \{m - \frac{n}{2} | m \in \mathbb{N}\}$, the formula $A_{i,j}(\lambda, \mu)$ simplifies due to the factorization of the underlying polynomial. Our experience suggests that the factorization indicates so called factorization identity, when a homomorphism of generalized Verma modules quotients through a homomorphism of generalized Verma modules of one of its summands (in the source) or a target homomorphism of generalized Verma modules. This naturally leads to the question of full composition structure of the branching problem going beyond the formulation in terms of the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}$.

Let us mention another interesting observation. The diagonal coefficients $A_{i,i}(\lambda, \mu) = A_{i,i}(\mu, \lambda)$ are, up to a rational multiple coming from the ratio of the product of Γ-functions, symmetric with respect to $\lambda \leftrightarrow \mu$. As a consequence, these polynomials belong to the algebra of $\mathbb{Z}_2$-invariants:

$$\mathbb{C}[\lambda, \mu]^{\mathbb{Z}_2} \xrightarrow{\sim} \mathbb{C}[\lambda\mu, \lambda + \mu].$$

Lemma 4.9 *The diagonal coefficients can be written as*

$$A_{i,i}(\lambda, \mu) = \frac{\Gamma(2i - N)}{2^{2i+1} i!\, \Gamma(-N)\Gamma(1 + i - \frac{n}{2} - \mu)\Gamma(1 + i - \frac{n}{2} - \lambda)} \cdot$$
$$(\Gamma(1 + i - \frac{n}{2} - \lambda)\Gamma(1 - \frac{n}{2} - \mu) {}_3F_2(-i, N - \lambda, 1 - N + \mu; 1, 1 - \frac{n}{2} - \lambda; 1) +$$
$$\Gamma(1 + i - \frac{n}{2} - \mu)\Gamma(1 - \frac{n}{2} - \lambda) {}_3F_2(-i, 1 - N + \lambda, N - \mu; 1, 1 - \frac{n}{2} - \mu; 1))$$

Proof:

It follows from the definition of ${}_3F_2$ that

$$\frac{1}{\Gamma(1 + i - \frac{n}{2} - \mu)\Gamma(1 + i - \frac{n}{2} - \lambda)} \cdot$$
$$(\Gamma(1 + i - \frac{n}{2} - \lambda)\Gamma(1 - \frac{n}{2} - \mu) {}_3F_2(-i, N - \lambda, 1 - N + \mu; 1, 1 - \frac{n}{2} - \lambda; 1) +$$
$$\Gamma(1 + i - \frac{n}{2} - \mu)\Gamma(1 - \frac{n}{2} - \lambda) {}_3F_2(-i, 1 - N + \lambda, N - \mu; 1, 1 - \frac{n}{2} - \mu; 1)) =$$
$$\sum_{m=0}^{i} (\frac{(-i)_m (N - \lambda)_m (1 - N + \mu)_m}{(1_m)(1 - \frac{n}{2} - \lambda)_m (1 - \frac{n}{2} - \mu)_{i-1}}$$
$$+ \frac{(-i)_m (N - \mu)_m (1 - N + \lambda)_m}{(1_m)(1 - \frac{n}{2} - \mu)_m (1 - \frac{n}{2} - \lambda)_{i-1}}) \frac{1}{m!}, \tag{51}$$

where the sum is finite due to the presence of $(-i)_m$ in the nominator. Using basic properties of the Pochhamer symbol, e.g. $(x)_m = (-)^m(-x + m - 1)_m$, an elementary manipulation yields the result.

□

Example 4.10 *As an example, in the case of* $i = 1$ *we have*

$$A_{1,1}(\lambda, \mu) = \frac{N(N-1)(\lambda\mu - N(\lambda + \mu) + (1 - \frac{n}{2} + N(N-1)))}{(2\lambda + n - 2)(2\mu + n - 2)} \tag{52}$$

Let us summarize our results in

Theorem 4.11 *Let $\mathfrak{g} = so(n+1,1,\mathbb{R})$ be a simple Lie algebra and $\mathfrak{p}$ its conformal parabolic subalgebra with commutative nilradical. Then the diagonal branching problem for the scalar generalized Verma $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})$-modules induced from character (λ,μ) is determined in the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$ by $\mathcal{U}(\mathfrak{g})$-isomorphism*

$$\mathcal{M}_{\lambda,\mu}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})|_{\mathfrak{g}} \simeq \bigoplus_{j=0}^{\infty}\mathcal{M}_{\lambda+\mu-2j}(\mathfrak{g},\mathfrak{p}). \tag{53}$$

Here the summand $\mathcal{M}_{\lambda+\mu-2j}(\mathfrak{g},\mathfrak{p})$ is generated by singular vector of homogeneity $2j$ and the form (25) with coefficients given by equation (42), $j\in\mathbb{N}$. In particular, the singular vectors are non-zero, unique up to a multiple, linearly independent and of expected weight (given by homogeneity), and the cardinality of the set of singular vectors is as predicted by Corollary 3.2.

The explicit formulas for the singular vectors will be given, in the dual language of bilinear differential operators, in the next section.

5 Application - the classification of bilinear conformal invariant differential operators on line bundles

Let M be a smooth (or complex) manifold equipped with the filtration of its tangent bundle

$$0\subset T^1M\subset\cdots\subset T^{m_0}M = TM,$$

$\mathcal{V}\to M$ a smooth (or holomorphic) vector bundle on M and $J^k\mathcal{V}\to M$ the weighted jet bundle over M, defined by

$$J^k\mathcal{V} = \bigcup_{x\in M} J^k_x\mathcal{V},\ J^k_x\mathcal{V}\xrightarrow{\sim}\oplus_{l=1}^{k} Hom(\mathcal{U}_l(gr(T_xM)),\Gamma(\mathcal{V}_x)),$$

where $\mathcal{U}_l(gr(T_xM))$ is the subspace of homogeneity k-elements in the universal enveloping algebra of the associated graded $gr(T_xM)$. A bilinear differential pairing between sections of the bundle $\mathcal{V}$ and sections of the bundle $\mathcal{W}$ to sections of the bundle $\mathcal{Y}$ is a vector bundle homomorphism

$$B : J^k\mathcal{V}\otimes J^l\mathcal{W}\to\mathcal{Y}. \tag{54}$$

In the case $\tilde{M} = \tilde{G}/\tilde{P}$ is a generalized flag manifold, a pairing is called invariant if it commutes with the action of $\tilde{G}$ on local sections of the homogeneous vector bundles $\mathcal{V},\mathcal{W},\mathcal{Y}$. Denoting $\mathbb{V}$, $\mathbb{W}$, $\mathbb{Y}$ the inducing $\tilde{P}$-representations of homogeneous vector bundles $\mathcal{V},\mathcal{W},\mathcal{Y}$, the space of $\tilde{G}$-invariant differential pairings can be algebraically characterized as

$$\begin{aligned}&((\mathcal{U}(\tilde{\mathfrak{g}})\otimes\mathcal{U}(\tilde{\mathfrak{g}}))\otimes_{\mathcal{U}(\tilde{\mathfrak{p}})\otimes\mathcal{U}(\tilde{\mathfrak{p}})} Hom(\mathbb{V}\otimes\mathbb{W},\mathbb{Y}))^{\tilde{P}}\simeq\\ &Hom_{\mathcal{U}(\tilde{\mathfrak{g}})}(\mathcal{M}(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}},\mathbb{Y}),\mathcal{M}(\tilde{\mathfrak{g}}\oplus\tilde{\mathfrak{g}},\tilde{\mathfrak{p}}\oplus\tilde{\mathfrak{p}},(\mathbb{V}^\vee\otimes\mathbb{W}^\vee)),\end{aligned} \tag{55}$$

where the superscript denotes the space of $\tilde{P}$-invariant elements and $\mathbb{V}^\vee$, $\mathbb{W}^\vee$ denote the dual representations, see e.g., [15]. In our case, we get

Theorem 5.1 *Let $G = SO_o(n+1,1,\mathbb{R})$ and P its conformal parabolic subgroup, $\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} | m \in \mathbb{N}\}$, $N \in \mathbb{N}$. Let us denote by $\mathcal{L}_{-\lambda}$ the homogeneous line bundle on n-dimensional sphere $G/P \simeq S^n$ induced from the character $\chi_{-\lambda}$ of P. Then there exists up to a multiple a unique set of bilinear conformally invariant operators*

$$B_N : C^\infty(G/P, \mathcal{L}_{-\lambda}) \times C^\infty(G/P, \mathcal{L}_{-\mu}) \to C^\infty(G/P, \mathcal{L}_{-\lambda-\mu-2N}) \tag{56}$$

of the form

$$B_N = \sum_{0 \leq i,j,k \leq N | i+j+k=N} A_{i,j}(\lambda, \mu) \tilde{s}^i \tilde{t}^j \tilde{r}^k, \tag{57}$$

where

$$\begin{aligned}
&A_{i,j}(\lambda, \mu) = \\
&\frac{\Gamma(i+j-N)\Gamma(1-\frac{n}{2}-\mu)\Gamma(1-i+j-N+\lambda)\Gamma(\lambda+\frac{n}{2}-i)}{2^{i+j}(-)^{i+j} i! j!\, \Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)\Gamma(\lambda+\frac{n}{2})} \cdot \\
&\sum_{k=0}^{i} (-)^k \binom{i}{k} (j-i+1+k)_{i-k} (\lambda + \frac{n}{2} - i)_{i-k} (\mu - N + 1)_k (\lambda - N + 1 - k)_k, \\
&A_{i,j}(\lambda, \mu) = A_{j,i}(\mu, \lambda),
\end{aligned} \tag{58}$$

such that

$$\tilde{s} = \sum_{i=1}^{n} \partial^2_{x_i} = \triangle_x,\ \tilde{t} = \sum_{i=1}^{n} \partial^2_{y_i} = \triangle_y,\ \tilde{r} = \sum_{i=1}^{n} \partial_{x_i} \partial_{y_i}. \tag{59}$$

Out of the range $\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} | m \in \mathbb{N}\}$, B_N is still an element of the previous set, but there might be additional ones indicating the emergence of a nontrivial composition structure.

Proof:

The proof is a direct consequence of Theorem 42 and duality (55), together with the application of inverse Fourier transform

$$x_j \longleftrightarrow -i\partial_{\xi_j},\ \partial_{x_j} \longleftrightarrow -i\xi_j$$

with $i \in \mathbb{C}$ the imaginary complex unit.

$\square$

In many applications, it is perhaps more convenient to express the bilinear differential operators in terms of tangent resp. normal coordinates $t_i = \frac{1}{2}(\xi_i + \nu_i)$ resp. $n_i = \frac{1}{2}(\xi_i - \nu_i)$, $i = 1, \dots, n$ to the diagonal submanifold, where

$$\begin{aligned}
r &= \frac{1}{4}(< t, t > - < n, n >), \\
s &= \frac{1}{4}(< t, t > + < n, n > +2 < t, n >), \\
t &= \frac{1}{4}(< t, t > + < n, n > -2 < t, n >).
\end{aligned} \tag{60}$$

6 Open problems and questions

It would be interesting to find the composition structure appearing for integral values of inducing representation-theoretical parameters, a question closely related to the factorization structure of special polynomials which are solutions of the problem. Another task is the extension of the present results to the case of vector valued representations, and also to the other couples $(\mathfrak{g}, \mathfrak{p})$ of a simple Lie algebra and its parabolic subalgebra of interest. This sounds clearly as a program rather than several unrelated questions.

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